Answer ALL questions from Section A.

All questions from Section B may be attempted, but only marks obtained on the best **two** *solutions from Section B will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

The following notation is used throughout: for any $a \in \mathbb{C}$ and $r > 0$, $S(a,r)$ denotes a positively oriented circular contour of radius r, centred at a; $D(a,r) = \{z : |z-a| < r\}$, $\overline{D}(a,r) = \{z : |z-a| \leq r\}$, $D'(a,r) = \{z : 0 < |z-a| < r\}$.

Section A

1. Find the Laurent expansion of $f(z) = \dfrac{1}{z^3 - z}$ in the following domains:

   (a) $0 < |z| < 1$,

   (b) $1 < |z-1| < 2$.

2. Let a function f be analytic on the punctured disk $D'(a,r)$ with some $a \in \mathbb{C}$ and $r > 0$. Describe three types of isolated singularities of the function f by explaining how they are related to the principal part of its Laurent expansion at the point a.

3. (a) Define the residue of a function having an isolated singularity at the point $a \in \mathbb{C}$.

   (b) Evaluate the residues of the function $f(z) = \frac{1}{\sin z}$ at the points $z = 0, \pi/2, \pi$.

4. Using an appropriate substitution and the Cauchy Integral Formula, evaluate the integral

$$\int_0^{2\pi} e^{e^{it}} dt.$$

5. Can one find an entire function g satisfying the property:

$$g\left(\frac{1}{n}\right) = g\left(-\frac{1}{n}\right) = \frac{1}{n^2}$$

for all $n = 1, 2, \ldots$? If yes, how many such functions are there? Justify your answer.

6. Determine all complex values z for which the series

$$\sum_{n=0}^{\infty} \left(\frac{z^n}{n!} + \frac{n^2}{z^n}\right)$$

converges.

Section B

7. (a) Suppose that f is holomorphic on the punctured disk $D'(a, r)$, $a \in \mathbb{C}, r > 0$, and that $|f(z)| \leq M$ for some constant $M > 0$ and all $z \in D'(a, r)$. Prove that a is a removable singularity of f.

   (b) Let g be an entire function such that $|g(z)| \geq 1$ for all $z \in \mathbb{C}$. Show that g is constant.

8. (a) Let g be an entire function. Show that one cannot have the inequality $|g^{(n)}(0)| \geq n^n n!$ for *all* $n = 1, 2, \ldots$.

   (b) Compute the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)^2} dx,$$

   where $a > 0$.

9. (a) Let f be an even entire function, i.e. $f(z) = f(-z)$ for all $z \in \mathbb{C}$. Show that the Taylor expansion of f at $a = 0$ contains only even powers of z.

   (b) Using an appropriate substitution and the Cauchy Residue Theorem, show that

$$\int_0^{2\pi} \frac{\cos\theta}{13 + 12\cos\theta} d\theta = -\frac{4\pi}{15}.$$

END OF PAPER